

Energy-Momentum Distribution of Non-Static Plane Symmetric Spacetimes in GR and TPT

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Abstract

This paper is devoted to explore the energy-momentum of non-static plane symmetric spacetimes in the context of General Relativity and teleparallel theory of gravity. For this purpose, we use four prescriptions, namely, Einstein, Landau-Lifshitz, Bergmann-Thomson and Møller in both theories. It is shown that the results for the first three prescriptions turn out to be same in both the theories but different for last prescription. It is mentioning here that our results coincide with the results obtained by Sharif and kanwal [1] for Bell-Szekeres metric under certain choice of the metric functions.

Keywords: Teleparallel Theory, Symmetric Plane, Energy.

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1 Introduction

Among all the available theories of gravitation in literature, the theory of General Relativity (GR) is considered as a standard theory of gravitation due to the fact that many physical aspects of nature have been verified experimentally in this theory. However, the problem of localization of energy and momentum in GR, is still most controversial [2]. A number of scientists have attempted to resolve this issue and gave their own definitions. As a pioneer, Einstein [3] gave an energy-momentum prescription for the localization of energy and momentum. Following him, many well known scientists like, Landau-Lifshitz [4], Møller [5], Bergmann-Thomson [6], Tolman [7] and Weinberg [8] gave their own energy-momentum prescriptions. To explore energy, the use of Cartesian coordinates are necessary for these prescriptions except Møller's prescription, which is independent of the coordinate system. Misner et, al. [2] proved that the energy can be localized in spherical coordinate system. After a short time, Cooperstok and Sarracino [9] proved that if the energy can be localized in spherical system then it can be localized in any other coordinate system. Virbhadra and his collaborators [10-12] explored the energy-momentum distribution of several spacetimes, such as, Kerr-Newmann, Kerr-Schild classes, Einstein-Rosen, Vaidya and Bonnor-Vaidya spacetimes. They showed that different energy-momentum prescriptions provide the same results which agree with those obtained by Penrose [13] and Tod [14] in the framework of quasi-local mass. Einstein [15] used the notion of tetrad field to unify gravitation and electromagnetism but he was not succeeded in his purpose. Hayashi and Nakano [16] formulated the tetrad theory of gravitation, which is known as teleparallel theory (TPT) of gravity or new General Relativity. This theory based on non-trivial tetrad fields and is defined on Weitzenböck [17] geometry. The curvature tensor of Weitzenböck connection vanishes identically but torsion remains non zero. In the frame work of TPT, gravitation is attributed to torsion [18] which plays the role of force while it geometrizes the underlying spacetime in the case of GR.

A number of scientists [19] hoped that the problem of localization of energy might be resolved in the frame work of TPT of gravity and results may coincide with those already existing in GR. Vargas [20] showed that total energy of the closed Friedmann-Robertson-Walker (FRW) universe is zero by using teleparallel (TP) version of Einstein and Landau-Lifshitz prescriptions. The results obtained by Vargas coincide with those found by Rozen

[21]. This opened the task for many authors who explored the energy momentum distribution of many spacetimes by using the TP version of different prescriptions. These prescriptions yield same results for some spacetimes and different in the case of others. Pereira et. al, [22] obtained the TP version of Schwarzschild and stationary axisymmetric Kerr solutions. Sharif and Jamil [23] found the TP versions of Friedmann models and Lewis-Papapetrou spacetimes and obtained interesting results. They [24] also explored the energy momentum distribution of the Lewis-Papapetrou spacetime by using the TP version of Møller prescription. They [25, 26] extended this work to stationary axisymmetric solutions of Einstein-Maxwell field equations and the Levi-Civita vacuum solutions. They [27, 28, 29] also explored the energy-momentum distribution of static axially symmetric, Friedmann models and the spatially homogenous rotating spacetimes by using different prescriptions in the context of TPT. Sharif and Kanwal [1] explored the energy-momentum distribution of the Bell-Szekeres metric in GR and TPT and showed that the four prescriptions ELLBTM yield same results in both the theories. Recently, Sharif and Sumaira [30] used Hamiltonian formulation of TEGR to explore the energy-momentum of Non-Vacuum Spacetimes and found consistent results.

The scheme of paper is as follows: in section 2 an over view of TPT is given. The different energy-momentum prescriptions in both GR and TPT are given in section 3. Section 4 is devoted to explore the energy-momentum distribution of non-static plane symmetric spacetimes in GR. The section 5, contains the energy-momentum distribution of non-static plane symmetric spacetimes in TPT. The last section furnishes the summary and discussion of the results obtained.

2 An Overview of the Teleparallel Theory

TPT is based on Weitzenböck connection given as [31]

$$\Gamma_{\mu\nu}^{\theta} = h_a^{\theta} \partial_{\nu} h^a_{\mu}, \quad (1)$$

where h_a^{ν} is a non-trivial tetrad. Its inverse field is denoted by h^a_{μ} and satisfy the relations

$$h^a_{\mu} h_a^{\nu} = \delta_{\mu}^{\nu}; \quad h^a_{\mu} h_b^{\mu} = \delta^a_b. \quad (2)$$

Here the Latin alphabet ($a, b, c, \dots = 0, 1, 2, 3$) are used to denote tangent space indices and the Greek alphabet ($\mu, \nu, \rho, \dots = 0, 1, 2, 3$) to denote space-time indices. The Riemannian metric in TPT arises as a product [31] of the tetrad field given by

$$g_{\mu\nu} = \eta_{ab} h^a{}_{\mu} h^b{}_{\nu}, \quad (3)$$

where η_{ab} is the Minkowski metric. For the Weitzenböck spacetime, the torsion is defined as [31]

$$T^{\theta}{}_{\mu\nu} = \Gamma^{\theta}{}_{\nu\mu} - \Gamma^{\theta}{}_{\mu\nu} \quad (4)$$

which is antisymmetric w.r.t. its last two indices. Due to the requirement of absolute parallelism, the curvature of the Weitzenböck connection vanishes identically. The Weitzenböck connection also satisfies the relation

$$\Gamma^{0\theta}{}_{\mu\nu} = \Gamma^{\theta}{}_{\mu\nu} - K^{\theta}{}_{\mu\nu}, \quad (5)$$

where

$$K^{\theta}{}_{\mu\nu} = \frac{1}{2} [T^{\theta}{}_{\mu}{}^{\nu} + T^{\theta}{}_{\nu}{}^{\mu} - T^{\theta}{}_{\mu\nu}] \quad (6)$$

is the **contortion tensor** and $\Gamma^{0\theta}{}_{\mu\nu}$ are the Christoffel symbols in GR.

3 Energy-Momentum Complexes

To explore the energy-momentum distribution of a given spacetime in the framework of GR, different approaches have been used by the different scientists. To derive an energy-momentum complex for the localization of energy and momentum is one of these approaches. The Einstein, Landau-Lifshitz, Bergmann-Thomson and Møller complexes in both GR and TPT are given as:

3.1 Energy-Momentum Complexes in GR

For **Einstein** prescription, the energy-momentum density components are given by [3]

$$\Theta_a^b = \frac{1}{16\pi} H^{bc}{}_{a,c}, \quad (7)$$

where $H^{bc}{}_a$ is a function of metric tensor and its first order derivatives given as

$$H^{bc}{}_a = \frac{g_{ad}}{\sqrt{-g}}[-g(g^{bd}g^{ce} - g^{cd}g^{be})]_{,e}. \quad (8)$$

Here Θ_0^0 stands for energy density and $\Theta_i^0 (i = 1, 2, 3)$ are the momentum density components and Θ_0^i are the current density components. The momentum four-vector is

$$P_a = \int_V \Theta_a^0 dx^1 dx^2 dx^3. \quad (9)$$

and the energy of the physical system is

$$P_0 = \int_V \Theta_0^0 dx^1 dx^2 dx^3. \quad (10)$$

It is mentioned here that these calculations are restricted to be done in Cartesian coordinates only to obtain physical results.

For **Landau-Lifshitz** prescription, the energy-momentum density components are given as [4]

$$L^{ab} = \frac{1}{16\pi} l^{abcd}{}_{,cd}, \quad (11)$$

where

$$l^{abcd} = (-g)(g^{ab}g^{cd} - g^{ac}g^{bd}). \quad (12)$$

The quantity L^{00} gives the energy density component of whole system and $L^{i0} (i = 1, 2, 3)$ represents the momentum density components.

For **Bergmann-Thomson** prescription, the energy-momentum density components are given by [6]

$$B^{ab} = \frac{1}{16\pi} M^{abc}{}_{,c}, \quad (13)$$

where

$$M^{abc} = g^{ad}V^{bc}{}_d \quad (14)$$

and

$$V^{bc}{}_d = \frac{g_{de}}{\sqrt{-g}}[-g(g^{be}g^{cf} - g^{ce}g^{bf})]_{,f}. \quad (15)$$

The quantity B^{00} represents energy density of the whole system and $B^{i0} (i = 1, 2, 3)$ represents the momentum density components.

Einstein, Landau-Lifshitz and Bergmann-Thomson energy-momentum prescriptions are coordinate dependent while **Møller** introduced another energy-momentum pseudo-tensor M_a^b which is coordinate independent, given as

$$M_a^b = \frac{1}{8\pi} K^{bc}{}_{a,c}, \quad (16)$$

where

$$K^{bc}{}_a = \sqrt{-g}(g_{ad,e} - g_{ae,d})g^{be}g^{cd}. \quad (17)$$

Clearly, $K^{bc}{}_a$ is antisymmetric w.r.t. its upper indices. M_0^0 is the energy density and $M_i^0 (i = 1, 2, 3)$ are the momentum density components and $M_0^i (i = 1, 2, 3)$ are the components of current density. The momentum four-vector is given by

$$p_a = \int \int_V \int M_a^0 dx^1 dx^2 dx^3, \quad (18)$$

where p_0 gives the energy and $p_i (i = 1, 2, 3)$ give the momentum. Using Gauss's theorem, the total energy-momentum components may be given in the form of surface integral as

$$p_a = \frac{1}{8\pi} \int_S \int K_a^{0c} \mathbf{n}_c \cdot d\mathbf{S}, \quad (19)$$

where \mathbf{n}_c is the outward unit normal vector over an infinitesimal surface element $d\mathbf{S}$.

3.2 Energy-Momentum Complexes in TPT

It was noticed that tetrad description of the gravitational field allows more satisfactory treatment of the gravitational energy-momentum. The Gauge field Lagrangian is given as

$$L = \frac{h}{16G\pi} \left[\frac{1}{4} F^a{}_{\mu\nu} F^b{}_{\theta\rho} g^{\mu\theta} N_{ab}{}^{\nu\rho} \right], \quad (20)$$

where $h = \det(h^a{}_\mu)$, G is the gravitational constant and $F^a{}_{\mu\nu}$ is field strength. In the presence of tetrad field, algebra and spacetime induces can be interchange and consequently it appears mixed up in the Lagrangian. It means that

$$N_{ab}{}^{\nu\rho} = \eta_{ab} g^{\nu\rho} = \eta_{ab} h_c^\nu h^{c\rho} \quad (21)$$

must now include all cyclic permutations of a, b and c. A simple calculation shows that

$$N_{ab}{}^{\nu\rho} = \eta_{ab} h_c^\nu h^{c\rho} + 2h_a{}^\rho h_b{}^\nu - 4h_a{}^\nu h_b{}^\rho. \quad (22)$$

Substituting Eq.(22) in Eq.(20), we get

$$L = \frac{h}{16G\pi} F^a{}_{\mu\nu} F^b{}_{\theta\rho} g^{\mu\theta} \left[\frac{1}{4} h_c^\nu h^{c\rho} \eta_{ab} + \frac{1}{2} h_a{}^\rho h_b{}^\nu - h_a{}^\nu h_b{}^\rho \right]. \quad (23)$$

Using the value of field strength $F^a{}_{\mu\nu} = c^2 h^a{}_\rho T^\rho{}_{\mu\nu}$ in Eq.(23), we have

$$\begin{aligned} L &= \frac{hc^4}{16G\pi} \left[\frac{1}{4} T^\rho{}_{\mu\nu} T^\mu{}_\rho{}^{\nu\mu} + \frac{1}{2} T^\rho{}_{\mu\nu} T^{\nu\mu}{}_\rho - T_{\rho\mu}{}^\rho T^{\nu\mu}{}_\nu \right], \\ &= \frac{hc^4}{16G\pi} S_\rho{}^{\mu\nu} T^\rho{}_{\mu\nu}, \\ L &= \frac{hc^4}{16G\pi} S^{\rho\mu\nu} T_{\rho\mu\nu}, \end{aligned}$$

where

$$S^{\rho\mu\nu} = \frac{1}{4} [T^{\rho\mu\nu} + T^{\mu\rho\nu} + T^{\nu\rho\mu}] + \frac{1}{2} [g^{\rho\mu\nu} T^{\theta\nu}{}_\theta - g^{\rho\nu} T^{\theta\mu}{}_\theta] \quad (24)$$

is a tensor written in terms of the Weitzenböck connection. Now, the Freud's superpotential is defined as

$$U_\rho{}^{\mu\nu} = h S_\rho{}^{\mu\nu}. \quad (25)$$

Vargas [20] gave TP version of Einstein, Bergmann-Thomson and Landau-Lifshitz prescriptions by using this superpotential as

$$hE_\nu^\mu = \frac{1}{4\pi} \partial_\lambda (U_\nu{}^{\mu\lambda}), \quad (26)$$

$$hL^{\mu\nu} = \frac{1}{4\pi} \partial_\lambda (h g^{\mu\beta} U_\beta{}^{\nu\lambda}), \quad (27)$$

$$hB^{\mu\nu} = \frac{1}{4\pi} \partial_\lambda (g^{\mu\beta} U_\beta{}^{\nu\lambda}). \quad (28)$$

The four-vector momentum for these complexes are given in the following relations.

$$p_\mu^E = \int_\Sigma h E_\mu^0 dx dy dz, \quad (29)$$

$$p_\mu^B = \int_\Sigma h B_\mu^0 dx dy dz, \quad (30)$$

$$p_\mu^L = \int_\Sigma h L_\mu^0 dx dy dz, \quad (31)$$

where p_0 and $p_i (i = 1, 2, 3)$ represent energy and momentum components respectively. In Eqs. (29) to (31), the integration is taken over the hypersurface Σ obtained by taking $t = \text{constant}$.

Now, we discuss Møller energy-momentum complex in the context of TPT. Mikhail et al. [19] defined the superpotential (which is antisymmetric in its last two indices) of the Møller tetrad theory as

$$U_\mu{}^{\nu\beta} = \frac{\sqrt{-g}}{2\kappa} P_{\chi\rho\sigma}^{\tau\nu\beta} [V^\rho g^{\sigma\chi} g_{\mu\tau} - \lambda g_{\tau\mu} K^{\chi\rho\sigma} - g_{\tau\mu} (1 - 2\lambda) K^{\sigma\rho\chi}], \quad (32)$$

where

$$P_{\chi\rho\sigma}^{\tau\nu\beta} = \delta_\chi^\tau g_{\rho\sigma}^{\nu\beta} + \delta_\rho^\tau g_{\sigma\chi}^{\nu\beta} - \delta_\sigma^\tau g_{\chi\rho}^{\nu\beta} \quad (33)$$

and $g_{\rho\sigma}^{\nu\beta}$ is a tensor quantity defined as

$$g_{\rho\sigma}^{\nu\beta} = \delta_\rho^\nu \delta_\sigma^\beta - \delta_\sigma^\nu \delta_\rho^\beta. \quad (34)$$

Here $K^{\sigma\rho\chi}$ is a contorsion tensor, g is the determinant of the metric tensor, κ is the coupling constant and V^μ is the basis vector field, which is given by

$$V_\mu = T^\nu{}_{\nu\mu}. \quad (35)$$

In TPT, the Møller's energy-momentum density is then defined as

$$\Xi_\mu^\nu = U_\mu{}^{\nu\rho}{}_{,\rho}, \quad (36)$$

where comma denotes ordinary differentiation. The energy E contained in a sphere of radius R is expressed by the volume integral as

$$E(R) = \int_{r=R} \Xi_0^0 dx dy dz, \quad (37)$$

and the spatial momentum $p_i, (i = 1, 2, 3)$ is given by

$$p_i(R) = \int_{r=R} \Xi_i^0 dx dy dz. \quad (38)$$

4 Energy-Momentum Distribution in GR

In this section, we explore the energy-momentum distribution of non-static plane symmetric spacetimes by using four different prescriptions of GR. The line element representing non-static plane symmetric spacetimes is given by [33]

$$ds^2 = e^{2\nu(t,x)} dt^2 - e^{2\mu(t,x)} dx^2 - e^{2\lambda(t,x)} (dy^2 + dz^2), \quad (39)$$

Making use of Eq.(39) in Eq.(8), we get the following non-vanishing components of H^{bc}_a

$$\begin{aligned} H^{01}_0 &= -H^{10}_0 &= -4e^{\nu-\mu+2\lambda}\lambda_x, \\ H^{01}_1 &= -H^{10}_1 &= -4e^{\mu-\nu+2\lambda}\lambda_t, \\ H^{02}_2 &= -H^{20}_2 &= -2e^{\nu-\mu+2\lambda}(\lambda_t + \mu_t), \\ H^{03}_3 &= -H^{30}_3 &= 2e^{\mu-\nu+2\lambda}(\lambda_t + \mu_t), \\ H^{12}_2 &= -H^{21}_2 &= 2e^{\nu-\mu+2\lambda}(\nu_x + \lambda_x) = H^{13}_3 = -H^{31}_3. \end{aligned} \quad (40)$$

Substituting Eq.(40) in Eq.(7), the non-zero energy-momentum density components of **Einstein's** prescription turn out to be

$$\Theta^{00} = -\frac{1}{4\pi}[\lambda_x(2\lambda_x - \mu_x + \nu_x) + \lambda_{xx}] e^{2\lambda-\nu-\mu}, \quad (41)$$

$$\Theta^{10} = \frac{1}{4\pi}[\lambda_t(2\lambda_x + \mu_x - \nu_x) + \lambda_{tx}] e^{2\lambda-\nu-\mu}. \quad (42)$$

Now, we substitute Eq.(39) in Eq.(12) and obtain the following non-vanishing components of l^{abcd} as

$$\begin{aligned} l^{0101} &= -l^{0110} = -l^{1001} = l^{1010} &= e^{4\lambda}, \\ l^{1313} &= -l^{1331} = -l^{3131} = l^{3113} &= -e^{2\nu+2\lambda}, \\ l^{0202} &= -l^{0220} = -l^{2002} = l^{2020} &= -e^{2\mu+2\lambda}, \\ l^{0303} &= -l^{0330} = -l^{3003} = l^{3030} &= e^{2\mu+2\lambda}, \\ l^{1212} &= -l^{1221} = -l^{2112} = l^{2121} &= e^{2\nu+2\lambda}, \\ l^{2323} &= -l^{2332} = -l^{3223} = l^{3232} &= -e^{2\nu+2\mu}. \end{aligned} \quad (43)$$

Making use of Eq.(43) in Eq.(11) yields the non-zero energy-momentum density components of **Landau-Lifshitz's** prescription as

$$L^{00} = -\frac{1}{4\pi}e^{4\lambda}(4\lambda_x^2 + \lambda_{xx}), \quad (44)$$

$$L^{10} = \frac{1}{4\pi}e^{4\lambda}(4\lambda_x\lambda_t + \lambda_{tx}). \quad (45)$$

Using Eq.(39) in Eq.(15), we get the following non-vanishing components of $V^{ab}{}_c$ as

$$\begin{aligned}
V^{10}{}_0 &= -V^{01}{}_0 = 4e^{2\lambda+\nu-\mu}\lambda_x, \\
V^{10}{}_1 &= -V^{01}{}_1 = 4e^{2\lambda-\nu+\mu}\lambda_t, \\
V^{20}{}_2 &= -V^{02}{}_2 = 2e^{2\lambda-\nu+\mu}(\lambda_t + \mu_t), \\
V^{30}{}_3 &= -V^{03}{}_3 = 2e^{2\lambda-\nu+\mu}(\lambda_t + \mu_t), \\
V^{31}{}_3 &= -V^{13}{}_3 = 2e^{2\lambda+\nu-\mu}(\lambda_x + \nu_x).
\end{aligned} \tag{46}$$

Substituting the values from Eq.(46) in Eq.(14) and then in Eq.(13), the non-zero energy-momentum density components of **Bergmann-Thomson's** prescription turn out to be

$$B^{00} = -\frac{1}{4\pi}[\lambda_x(2\lambda_x - \mu_x - \nu_x) + \lambda_{xx}] e^{2\lambda-\nu-\mu} \tag{47}$$

$$B^{10} = \frac{1}{4\pi}[\lambda_t(2\lambda_x - \mu_x - \nu_x) + \lambda_{tx}] e^{2\lambda-\nu-\mu} \tag{48}$$

The non-vanishing components of $K^{ab}{}_c$ are obtained by using Eq.(39) in Eq.(17) as

$$\begin{aligned}
K^{01}{}_0 &= -K^{10}{}_0 = 2e^{\nu-\mu+2\lambda}\nu_x, \\
K^{01}{}_1 &= -K^{10}{}_1 = 2e^{\mu-\nu+2\lambda}\mu_t, \\
K^{02}{}_2 &= -K^{20}{}_2 = 2e^{\mu-\nu+2\lambda}\lambda_t, \\
K^{03}{}_3 &= -K^{30}{}_3 = 2e^{\mu-\nu+2\lambda}\lambda_t.
\end{aligned} \tag{49}$$

In view of Eq.(49), the non-zero energy-momentum density components of **Møller's** prescription in contravariant form are obtained from Eq.(16) as

$$M^{00} = \frac{1}{4\pi}[\nu_x(2\lambda_x - \mu_x + \nu_x) + \nu_{xx}] e^{2\lambda-\nu-\mu}, \tag{50}$$

$$M^{10} = -\frac{1}{4\pi}[\mu_t(2\lambda_x + \mu_x - \nu_x) + \mu_{tx}] e^{2\lambda-\nu-\mu}. \tag{51}$$

5 Energy-Momentum Distribution in TPT

In this section, we use the above mentioned four prescriptions in the context of TPT to evaluate the energy-momentum distribution of non-static plane

symmetric spacetimes. The corresponding tetrad components of the metric (39) are given as

$$h^a{}_\mu = \begin{pmatrix} e^{\nu(t,x)} & 0 & 0 & 0 \\ 0 & e^{\mu(t,x)} & 0 & 0 \\ 0 & 0 & e^{\lambda(t,x)} & 0 \\ 0 & 0 & 0 & e^{\lambda(t,x)} \end{pmatrix} \quad (52)$$

with its inverse

$$h_a{}^\mu = \begin{pmatrix} e^{-\nu(t,x)} & 0 & 0 & 0 \\ 0 & e^{-\mu(t,x)} & 0 & 0 \\ 0 & 0 & e^{-\lambda(t,x)} & 0 \\ 0 & 0 & 0 & e^{-\lambda(t,x)} \end{pmatrix}. \quad (53)$$

One can easily verify Eqs.(2) and (3) with the help of Eqs.(52) and (53). Substituting Eqs.(52) and (53) in Eq.(1), we get the following non-zero components of Weitzenböck connection

$$\begin{aligned} \Gamma^0_{00} &= \nu_t, \\ \Gamma^0_{01} &= \nu_x, \\ \Gamma^1_{10} &= \mu_t, \\ \Gamma^1_{11} &= \mu_x, \\ \Gamma^2_{20} = \Gamma^3_{30} &= \lambda_t, \\ \Gamma^2_{21} = \Gamma^3_{31} &= \lambda_x. \end{aligned} \quad (54)$$

Eq.(4) then gives the corresponding non-vanishing components of the torsion tensor as

$$\begin{aligned} T^0_{10} = -T^0_{01} &= \nu_t, \\ T^1_{01} = -T^1_{10} &= \mu_t, \\ T^2_{02} = -T^2_{20} &= \lambda_t, \\ T^2_{21} = -T^2_{12} &= \lambda_x \\ T^3_{03} = -T^3_{30} &= \lambda_t, \\ T^3_{13} = -T^3_{31} &= \lambda_x. \end{aligned} \quad (55)$$

Multiplying the above components of the torsion tensor with relevant $g^{\mu\nu}$ and then using in Eq.(24), we get the following non-zero components of the

tensor S^{abc} as

$$\begin{aligned}
S^{010} = -S^{001} &= e^{-2(\nu+\mu)} \lambda_x, \\
S^{101} = -S^{110} &= e^{-2(\mu+\nu)} \lambda_t, \\
S^{202} = -S^{220} &= \frac{1}{2} e^{-2(\lambda+\nu)} (\lambda_t + \mu_t), \\
S^{221} = -S^{212} &= \frac{1}{2} e^{-2(\lambda+\mu)} (\lambda_x + \nu_x), \\
S^{303} = -S^{330} &= \frac{1}{2} e^{-2(\lambda+\nu)} (\lambda_t + \mu_t), \\
S^{313} = -S^{331} &= \frac{1}{2} e^{-2(\lambda+\mu)} (\lambda_x + \nu_x).
\end{aligned} \tag{56}$$

Making use of Eq.(56) in Eq.(25) yields the following non-zero components of the superpotential as

$$\begin{aligned}
U_0^{10} = -U_0^{01} &= e^{2\lambda+\nu+\mu} \lambda_x, \\
U_1^{10} = -U_1^{01} &= e^{2\lambda+\mu-\nu} \lambda_t, \\
U_2^{20} = -U_2^{02} &= \frac{1}{2} e^{2\lambda+\mu-\nu} (\lambda_t + \mu_t), \\
U_2^{12} = -U_2^{21} &= \frac{1}{2} e^{2\lambda-\mu+\nu} (\lambda_x + \nu_x), \\
U_3^{30} = -U_3^{03} &= \frac{1}{2} e^{2\lambda+\mu-\nu} (\lambda_t + \mu_t), \\
U_3^{13} = -U_3^{31} &= \frac{1}{2} e^{2\lambda-\mu+\nu} (\lambda_x + \nu_x).
\end{aligned} \tag{57}$$

Using Eq.(57) in Eqs.(26), (27) and (28), the non-vanishing energy-momentum density components of Einstein, Landau-Lifshitz and Bergmann-Thomson prescriptions respectively are

$$hE^{00} = -\frac{1}{4\pi} [\lambda_x(2\lambda_x - \mu_x + \nu_x) + \lambda_{xx}] e^{2\lambda-\nu-\mu}, \tag{58}$$

$$hE^{10} = \frac{1}{4\pi} [\lambda_t(2\lambda_x + \mu_x - \nu_x) + \lambda_{tx}] e^{2\lambda-\nu-\mu}, \tag{59}$$

$$hL^{00} = -\frac{1}{4\pi} e^{4\lambda} (4\lambda_x^2 + \lambda_{xx}), \tag{60}$$

$$hL^{10} = \frac{1}{4\pi} e^{4\lambda} (4\lambda_x \lambda_t + \lambda_{tx}) \tag{61}$$

and

$$hB^{00} = -\frac{1}{4\pi}[\lambda_x(2\lambda_x - \mu_x - \nu_x) + \lambda_{xx}] e^{2\lambda-\nu-\mu}, \quad (62)$$

$$hB^{10} = \frac{1}{4\pi}[\lambda_t(2\lambda_x - \mu_x - \nu_x) + \lambda_{tx}] e^{2\lambda-\nu-\mu}. \quad (63)$$

Now, using Eq.(55) in Eq.(6) and then multiplying by relevant components of $g^{\mu\nu}$, we get the following non-vanishing components of the contorsion tensor in contravariant form as

$$\begin{aligned} K^{010} = -K^{100} &= e^{-2(\mu+\nu)}\nu_x, \\ K^{101} = -K^{011} &= e^{-2(\mu+\nu)}\mu_t, \\ K^{202} = -K^{022} &= e^{-2(\lambda+\nu)}\lambda_t, \\ K^{122} = -K^{212} &= e^{-2(\lambda+\mu)}\lambda_x, \\ K^{303} = -K^{033} &= e^{-2(\lambda+\nu)}\lambda_t, \\ K^{313} = -K^{133} &= e^{-2(\lambda+\mu)}\lambda_x. \end{aligned} \quad (64)$$

Clearly, the contorsion tensor is antisymmetric w.r.t. its first two indices. Substituting Eq.(55) in Eq.(35) and then multiplying by relevant components of $g^{\mu\nu}$ yields the non-zero basic vector components in contravariant form as

$$\begin{aligned} V^0 &= -e^{-2\nu}(\mu_t + 2\lambda_t), \\ V^1 &= e^{-2\mu}(\nu_x + 2\lambda_x). \end{aligned} \quad (65)$$

Substituting Eqs.(39), (64), (65) and $\kappa = 8\pi$ (taking, $G = c = 1$) in Eq.(32), the required non-vanishing components of the superpotential turn out to be

$$\begin{aligned} U_0^{01} &= -\frac{1}{4\pi}\lambda_x e^{2\lambda-\mu+\nu}, \\ U_1^{01} &= -\frac{1}{4\pi}\lambda_t e^{2\lambda+\mu-\nu}. \end{aligned} \quad (66)$$

In view of Eq.(66), the non-zero energy-momentum density components in contravariant form can be obtained from Eq.(36) after multiplication with g^{00} and g^{11} as

$$\begin{aligned} \Xi^{00} &= -\frac{1}{4\pi}[\lambda_x(2\lambda_x - \mu_x + \nu_x) + \lambda_{xx}] e^{2\lambda-\mu-\nu}, \\ \Xi^{10} &= \frac{1}{4\pi}[\lambda_t(2\lambda_x + \mu_x - \nu_x) + \lambda_{tx}] e^{2\lambda-\mu-\nu}. \end{aligned} \quad (67)$$

6 Summary and Discussion

Energy-momentum is an important conserved quantity whose definition has been under investigation since the birth of GR. Although, the problem of localization of energy is unresolved and controversial but much attention has been given by different scientists to resolve it. Here, we have discussed the problem of localization of energy-momentum in two different frameworks of GR and TPT by using different energy-momentum complexes. We used Einstein, Landau-Lifshitz, Bergmann-Thomson and Møller prescriptions to explore the energy-momentum distribution of non-static plane symmetric spacetimes in the context of both GR and TPT. Although, on the basis of this work we are not able to resolve the longstanding and crucial problem of the localization of energy but it adds one more example which may be used to make a conjuncture about the localization of energy at some stage. The results obtained so far are given in the following tables (1-8):

Table 1. Energy-Momentum Density (EMD) Components of Einstein's Prescription in GR

EMD	Expressions
Θ^{00}	$-\frac{1}{4\pi}[\lambda_x(2\lambda_x - \mu_x + \nu_x) + \lambda_{xx}] e^{2\lambda - \mu - \nu}$
Θ^{10}	$\frac{1}{4\pi}[\lambda_t(2\lambda_x + \mu_x - \nu_x) + \lambda_{tx}] e^{2\lambda - \mu - \nu}$

Table 2. Energy-Momentum Density (EMD) Components of Einstein's Prescription in TPT

EMD	Expressions
hE^{00}	$-\frac{1}{4\pi}[\lambda_x(2\lambda_x - \mu_x + \nu_x) + \lambda_{xx}] e^{2\lambda - \mu - \nu}$
hE^{10}	$\frac{1}{4\pi}[\lambda_t(2\lambda_x + \mu_x - \nu_x) + \lambda_{tx}] e^{2\lambda - \mu - \nu}$

Table 3. Energy-Momentum Density (EMD) Components of Landau-Lifshitz's Prescription in GR

EMD	Expressions
L^{00}	$-\frac{1}{4\pi}(4\lambda_x^2 + \lambda_{xx}) e^{4\lambda}$
L^{10}	$\frac{1}{4\pi}(4\lambda_x\lambda_t + \lambda_{tx}) e^{4\lambda}$

Table 4. Energy-Momentum Density (EMD) Components of Landau-Lifshitz's Prescription in TPT

EMD	Expressions
hL^{00}	$-\frac{1}{4\pi}(4\lambda_x^2 + \lambda_{xx}) e^{4\lambda}$
hL^{10}	$\frac{1}{4\pi}(4\lambda_x\lambda_t + \lambda_{tx}) e^{4\lambda}$

Table 5. Energy-Momentum Density (EMD) Components of Bergmann-Thamson's Prescription in GR

EMD	Expressions
B^{00}	$-\frac{1}{4\pi}[\lambda_x(2\lambda_x - \nu_x - \mu_x) + \lambda_{xx}] e^{2\lambda-\mu-\nu}$
B^{10}	$\frac{1}{4\pi}[\lambda_t(2\lambda_x - \mu_x - \nu_x) + \lambda_{tx}] e^{2\lambda-\mu-\nu}$

Table 6. Energy-Momentum Density (EMD) Components of Bergmann-Thamson's Prescription in TPT

EMD	Expressions
hB^{00}	$-\frac{1}{4\pi}[\lambda_x(2\lambda_x - \nu_x - \mu_x) + \lambda_{xx}] e^{2\lambda-\mu-\nu}$
hB^{10}	$\frac{1}{4\pi}[\lambda_t(2\lambda_x - \mu_x - \nu_x) + \lambda_{tx}] e^{2\lambda-\mu-\nu}$

Table 7. Energy-Momentum Density (EMD) Components of Møller's Prescription in GR

EMD	Expressions
M^{00}	$\frac{1}{4\pi}[\nu_x(2\lambda_x - \mu_x + \nu_x) + \nu_{xx}] e^{2\lambda-\mu+\nu}$
M^{10}	$-\frac{1}{4\pi}[\mu_t(2\lambda_x + \mu_x - \nu_x) + \mu_{tx}] e^{2\lambda-\mu+\nu}$

Table 8. Energy-Momentum(E-M) densities in Møller's Prescription in TPT

EMD	Expressions
Ξ^{00}	$-\frac{1}{4\pi}[\lambda_x(2\lambda_x - \mu_x + \nu_x) + \lambda_{xx}] e^{2\lambda-\mu-\nu}$
Ξ^{10}	$\frac{1}{4\pi}[\lambda_t(2\lambda_x + \mu_x - \nu_x) + \lambda_{tx}] e^{2\lambda-\mu-\nu}$

These tables show that the energy-momentum density components turn out to be well defined and finite for each prescription in both GR and TPT. It is mentioning here that the only non-vanishing component of the momentum density is along x -axis while the other components turn out to be zero in each case. It is due to the fact that we have considered the metric in which the metric functions are depending on t and x only. Also, we can obtain the corresponding results along y - or z -axes by considering the metric function depending on y or z along with t . From the results given in tables 1 – 6, it is noted that the three prescriptions, namely, Einstein, Landau-Lifshitz and Bergmann-Thomson (ELLBT) yield the same energy-momentum distribution of non-static plane symmetric spacetimes in both GR and TPT, while the results of Møller's prescriptions in both the theories are different. Further, tables 2, 8 show that the energy as well as momentum density components of Einstein's and Møller's prescriptions turn out to be same in TPT. It is worth mentioning here that our results coincide with the results obtained by Sharif and Kanwal [1] for Bell-Szekeres metric under certain choice of the metric functions.

In the end, it is suggested that the issue of localization of energy may be tackled in some other theories, like $f(r)$ theory of gravity.

Acknowledgment

We acknowledge the enabling role of the Higher Education Commission Islamabad, Pakistan, and appreciate its financial support through the *Indigenous PhD 5000 Fellowship Program Batch-III*.

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